國 立 臺 北 科 技 大 學

九十八學年第一學期電機系博士班資格考試

隨機程序試題

注意事項：

1. 本試題共【5】題，配分共 100 分。
2. 請按順序標明題號作答，不必抄題。
3. 全部答案均須答在試卷答案欄內，否則不予計分。
4. 考試時間:二小時。

1. (a) (6pt) Give the definition of a probability measure P on a measurable space $(\Omega, \boldsymbol{F})$.
   (b) (6pt) Use the definition in part (a). Let F, G be two events, show that $P(F \cup G) = P(F) + P(G) - P(F \cap G)$.
   (c) (8pt) Let Q be a set function defined on the measurable space $((0,1], B((0,1]))$ where $B((0,1]$ is the Borel field on the unit interval $(0,1]$ and Q is defining by $Q(A) = \sum_{i=1}^{\infty} c2^{-i} I_{\{2^{-i}\}}(A)$ where $I_{\{x\}}(B) = \begin{cases} 1 & x \in B \\ 0 & x \not\in B \end{cases}$ and c is a constant. Is possible Q be a probability measure on $((0,1], B((0,1]))$. Explain your answer.

2. Let $X_1, X_2, \cdots$ be i.i.d. nonnegative random variables with exponential probability density function $f_X(x) = e^{-x} I_{[0,\infty)}(x)$. Let $Y_n = \max\{X_1, \cdots, X_n\}$.
   (a) (6pt) For a fixed $a > 0,$ compute $P(Y_n > a,)$.
   (b) (6pt) If $0 < a < b,$ compute $P(Y_n \le a, Y_{n+1} > b)$.
   (c) (8pt) Fix $a > 0$, and for $i \ge 1$ let A_i be the event that $X_i > a$. For a given $m \in \mathbb{N}$, Find the probability that none of the events $A_m, A_{m+1}, \cdots$ occurs. What is the probability that at least one of the events $A_m, A_{m+1}, \cdots$ occurs.

3. (a) (10pt) Prove the **Weak Law of Large number**: Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of uncorrelated random variables with common means $EX_i = u$ and $\lim_n \frac{1}{n^2} \sum_{i=1}^{n} \mathrm{Var} X_i = 0$. Show that $S_n = \frac{X_1 + X_2 + \cdots + X_n}{n} \xrightarrow{P} u$.

(b) (10pt) Prove the **Cauchy-Schwarz inequality**: X, Y are square-integrable *r.v.*'s, then $(E(XY))^2 \leq E(X^2)E(Y^2)$ with equality if and only if $X + \lambda Y = 0$.

4. (a) (5pt) Give the definition of an independent increment process.
   (b) (5pt) Give an example of independent increment process and verify it.
   (c) (5pt) Give the definition of a stationary process.
   (d) (5pt) Give the definition of an random sequence $(X_n)_{n \in \square}$ converges almost surely (with probability one).

5. Let $\{X_n; n = 0,1,2,\cdots\}$ be a i.i.d. random process with alphabet $\{0,1\}$ and $P(X_n = 1) = \rho > 0$. Define the random process $\{W_n; n \in \square\}$ by $W_n = X_n \oplus X_{n-1}$ where $\oplus$ is mod 2 addition(i.e. $W_n = \begin{cases} 1 & X_n \neq X_{n-1} \\ 0 & X_n = X_{n-1} \end{cases}$.)

   (a) (6pt) Find the pmf $p_{W_n}(k)$ of W_n.
   (b) (6pt) Find the Correlation function $R_W(i, j)$ of $\{W_n; n \in \square\}$.
   (c) (8pt) Show that $\{W_n; n \in \square\}$ is a wide-sense stationary process and find its power spectral density $S_W(f)$.