國 立 臺 北 科 技 大 學

一百學年第二學期電機系博士班資格考試

隨機程序試題

注意事項：

1. 本試題共【5】題，配分共 100 分。
2. 請按順序標明題號作答，不必抄題。
3. 全部答案均須答在試卷答案欄內，否則不予計分。
4. 考試時間:二小時。

1. (a) (5pt) Give the definition of a wise-sense stationary process.
   (b) (5pt) Give the definition of a random variable X on a probability space $(\Omega, \boldsymbol{F}, P)$.
   (c) (5pt) Give the definition of a random process is a counting process.
   (d) (5pt) Give the definition of the σ-field generated by a random variable X.

2. (a) (9pt) Give the definitions for a random sequence $\{X_n; n \in \mathbb{N}\}$ converging in the following modes: point wise convergence, almost sure convergence, and convergence in r_th mean.
   (b) $\{X_n; n \in \mathbb{N}\}$ is a sequence of $r.v$'s on $((0,1], B(0,1], P)$ where P is the Lebesque measure on unit interval. Let $X_n(\omega) = \begin{cases} n^{1/5} & \frac{n-1}{n} < \omega \le 1 \\ 0 & \text{else} \end{cases}$.
   (i) (3pt) Show that $\{X_n; n \in \mathbb{N}\}$ converges almost surely, but not point-wisely.
   (ii) (3pt) Show that $\{X_n; n \in \mathbb{N}\}$ converges in r_th mean for $r \le 4$.
   (c) (5pt) Show that if $\{X_n; n \in \mathbb{N}\}$ converges in k_th mean for some $k \in \mathbb{N}$, then $\{X_n; n \in \mathbb{N}\}$ also converges in s_th mean where $s < k$ and $s \in \mathbb{N}$.

3. (a) (10pt) **Gram-Schmidt orthogonalization procedure**: Let $\overline{X}$ be a n-dimensional random vector with mean vector $\overline{m} = [m_1 \ \cdots \ m_n]^T$ and covariance matrix $C_{\overline{X}}$. Define the random vector $\overline{Z} = [Z_1, \cdots, Z_n]^T$ by $Z_1 = X_1 - m_1$,

$Z_{k+1} = X_{k+1} - m_{k+1} - \sum_{j=1}^{k} \frac{\operatorname{cov}(X_{k+1}, Z_j)}{\operatorname{var}(Z_j)} Z_j$ for $k = 1, \cdots, n-1$, where the fraction $0/0$ is defined as 0. Show that $\bar{Z}$ is a 0-mean uncorrelated random vector, and there exists a lower triangular A such that $\bar{X} = A\bar{Z} + \bar{m}$.

(b) (10pt) Let $\bar{m} = [1\ 2\ 3]^T$ and $C_{\bar{X}} = \begin{bmatrix} 6 & 3 & 1 \\ 3 & 4 & -1 \\ 1 & -1 & 2 \end{bmatrix}$. Find A in part (a).

4. Let $\{X_n; n = 1, 2, \ldots\}$ be a random process such that $X_n = \cos(nY)$ where Y is a random variable with uniform density on $(-\pi, \pi)$.
   (a) (4pt) Find EY and characteristic function $\phi_Y(u)$ of Y
   (b) (9pt) Find EX_n, covariance function $k_X(m,n)$ and $E[X_n^2 X_m]$ for $\{X_n; n = 1, 2, \ldots\}$.
   (c) (4pt) Is $\{X_n; n = 1, 2, \ldots\}$ wise-sense stationary? Explain your answer.
   (d) (3pt) Is $\{X_n; n = 1, 2, \ldots\}$ stationary? Explain your answer.

5. Let $\{X_n; n \in \mathbb{Z}\}$ be an i.i.d. process with pdf $f_{X_n}(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$ and let $\{N_t, t \geq 0\}$ be the Poisson counting process with rate $\lambda = 2$. Define $\{Y_t, t \geq 0\}$ by $Y_t = \sum_{k=1}^{N_t} X_k$.
   (a) (4pt) Show that the characteristic function of X_n is $\Phi_{X_n}(u) = \dfrac{1}{1 - iu}$.
   (b) (4pt) Find EY_t.
   (c) (6pt) Find $\operatorname{var}(Y_t)$
   (d) (6pt) Find the characteristic function $\phi_{Y_t}(u)$ of Y_t.